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On the difference equation $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$

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Abstract

In this paper we study the behavior of the solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots$$

where α is a negative number. Included are results which considerably improve and correct those in the recently published paper: [A.E. Hamza, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl. 322 (2006), 668–674]. We also refute Conjecture 2 in [G. Ladas, Open problems and conjectures, J. Difference. Equ. Appl. 7 (2) (2001), 477–482].

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1. Introduction

Recently there has been a great interest in studying nonlinear and rational difference equations (c.f. [1–44] and the references therein). One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economics, probability theory, genetics, psychology, sociology etc. Such equations also appear naturally as discrete analogues of differential equations which model various biological and economic systems (see, for example, [10,15,20,22,24,26,29] and the references therein).

In [12] Hamza investigates the behavior of solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0 := 0, 1, \ldots, \tag{1}$$

where $\alpha < -1$. Eq. (1) and its generalizations for $\alpha < 0$, were also considered in [2,8,13]. Positive solutions of Eq. (1) for the case $\alpha > 0$, and some generalizations of the equation, were considered in [9,18,30,31,34,40,41]. See, also [11, 42,44] for closely related results. In [12] the author proved the following results (summarized in a theorem).

Theorem A. *Consider Eq. (1). Then the following statements are true.*

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(a) *The equilibrium point* $\bar{x} = \alpha + 1$ *is locally asymptotically stable if* $\alpha < -2$, *and unstable if* $\alpha \in (-2, -1) \cup (-1, 0)$.
(b) *If there are* $m, M \in (0, 1)$ *such that* $m \le M$ *and*

$$\frac{-m}{M(1-M)} \le \alpha \le \frac{-M}{m(1-m)}, \tag{2}$$

then $\bar{x}$ *is global attractor relative to the set* $\cup_{k\ge 1}[k\alpha M, k\alpha m]^2$.
(c) *If* $\alpha < -1$, *and* (x_n) *is a nontrivial solution of Eq.* (1), *such that* $x_n \ge \alpha + 1$ *for every* $n \ge n_0$, *then it converges monotonically to zero.*
(d) *If* $\alpha < -1$, *and* (x_n) *is a nontrivial solution of Eq.* (1), *such that* $x_n \le \alpha + 1$ *for every* $n \ge n_0$, *then it decreases to* $-\infty$.

The first result in Theorem A is a simple consequence of the linearized stability theorem (see, for example, [22]).

The second result is not correct. Namely, the author used the following theorem in the proof of the result, which he attributed to himself:

Theorem B. *Let* $[a, b]$ *be an invariant interval under a continuous function* $f(x, y)$ *which is non-increasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$. *Assume that* $\bar{x} \in [a, b]$ *is a unique equilibrium point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0. \tag{3}$$

If the system

$$x = f(y, x) \quad \text{and} \quad y = f(x, y) \tag{4}$$

has exactly one solution in $[a, b]^2$, *then* $\bar{x}$ *is a global attractor with basin* $[a, b]^2$.

However, if system (4) has exactly one solution in $[a, b]^2$ it must be $x = y = \bar{x}$, which means that Eq. (3) has no solutions of prime period two in $[a, b]$. From this, it follows that Theorem B is a special case of Theorem 1.4.6 in [22]. On the other hand, Theorem B should have been applied to the function

$$f(x, y) = \alpha + \frac{y}{x}$$

where $x, y \in [\alpha M, \alpha m]$, with some $m, M \in (0, 1)$ and $\alpha < -1$, which is non-decreasing in $x \in [\alpha M, \alpha m]$ for each $y \in [\alpha M, \alpha m]$, and is non-increasing in $y \in [\alpha M, \alpha m]$ for each $x \in [\alpha M, \alpha m]$. Hence, the author applied a wrong theorem to Eq. (1). The mistake was made since he claims that $f(x, y) = \alpha + \frac{x}{y}$, for the case of Eq. (1) which is not true. In fact, this f relates to the difference equation

$$x_{n+1} = \alpha + \frac{x_n}{x_{n-1}}.$$

The invariant intervals for Eq. (1) under condition (2) are correct, but they seems a little bit artificial.

Finally, the results in (c) and (d) are quite simple. Namely, if (x_n) is a nontrivial solution of Eq. (1) so that there is $n_0 \in \mathbb{N}$ such that $x_n \ge \alpha + 1$ for $n \ge n_0$, then $\alpha + 1 \le x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, for every $n \ge n_0 - 1$. If $x_{n_0-1} > 0$ this implies that $0 < x_{n_0} \le x_{n_0-1}$, and by induction we have that $0 < \cdots \le x_{n_0+1} \le x_{n_0} \le x_{n_0-1}$. Similarly if $x_{n_0-1} < 0$ then $\alpha + 1 \le x_{n_0-1} \le x_{n_0} \le \cdots < 0$. Hence, the sequence (x_n) is convergent and since it cannot converge to $\alpha + 1$ (here we use the condition that (x_n) is nontrivial) its limit must be equal to zero. On the other hand, if $x_n \le \alpha + 1$ for every $n \ge n_0$, then we have $\alpha + 1 \ge x_{n_0+2} = \alpha + \frac{x_{n_0}}{x_{n_0+1}}$, which implies that $x_{n_0+1} \le x_{n_0} \le \alpha + 1$. Repeating this procedure we obtain

$$\cdots \le x_{n_0+2} \le x_{n_0+1} \le x_{n_0} \le \alpha + 1.$$

From this and since (x_n) is nontrivial solution of Eq. (1), it follows that $\lim_{n\to\infty} x_n = -\infty$.

Remark 1. Note that the proof of the statements (c) and (d) in Theorem A hold also for the case $\alpha = -1$.

However, many other much more interesting and harder problems are not treated in [12] at all. Here are some of them:

Question 1. Is there a solution of Eq. (1) converging to zero?

Question 2. If there is a positive solution of Eq. (1) converging to zero, is it possible to find its asymptotics?

Question 3. Is there a solution of Eq. (1) which decreasingly tends to $-\infty$?

Question 4. Is there a solution of Eq. (1) which is negative?

Question 5. What can we say about stability of Eq. (1) for the case $\alpha = -2$?

Our aim here is to give some answers to these questions and extend some of the results in [12].

The paper is organized as follows. Section 2 is devoted to the study of the monotone solutions of Eq. (1) and some closely related equations. In Section 3 we show that, for the case $\alpha = -1$, there are solutions of Eq. (1) which decreasingly converge to zero and we find their asymptotics, moreover, we show that every positive solution of Eq. (1) converges to zero. The case $\alpha = -2$ is treated in Section 4. In Section 5 we show that there are no negative decreasing solutions of Eq. (1). Natural invariant intervals for Eq. (1), in the case $\alpha < -3$ are found in Section 6, as well as we prove a global convergence result. In Section 7, we apply results from previous sections and refute a conjecture posed by Ladas in [23].

2. Solutions converging to zero—how to find them

First, we address Question 1, that is, the existence of solutions of Eq. (1) converging to zero. For the results devoted to the area see the following papers [2,4,8,14,19,28,36,43] and the references therein.

A general result which can help in proving the existence of monotone solutions (also in the non-autonomous case) were developed by the present author in [36] (in order to prove Open Problem 11.4.10 in [22]), based on L. Berg's ideas in [4] which use asymptotics. We have proved the following inclusion theorem.

Theorem C. *Let $f : I^{k+1} \to I$ be a continuous and non-decreasing function in each argument on the interval $I \subset \mathbb{R}$, and let (y_n) and (z_n) be sequences such that $y_n < z_n$ for $n \geq n_0$ and*

$$y_{n-k} \leq f(y_{n-k+1}, \dots, y_{n+1}), \qquad f(z_{n-k+1}, \dots, z_{n+1}) \leq z_{n-k}, \quad \text{for } n > n_0 + k - 1. \tag{5}$$

Then, the difference equation

$$x_{n-k} = f(x_{n-k+1}, \dots, x_{n+1}), \tag{6}$$

has a solution such that

$$y_n \leq x_n \leq z_n, \quad \text{for } n \geq n_0. \tag{7}$$

Asymptotics for solutions of difference equations have been investigated by Berg and Stević, see, for example, [2–7,11,25–27,29,33,35–39,41] and the reference therein. Some methods for the construction of the bounds (y_n) and (z_n) can be found in [3–6].

We will use this opportunity to describe briefly a Berg–Stević method which can be used in proving that a difference equation has monotonic solutions. For a difference equation with the equilibrium $\bar{x}$ we consider its linearized equation about the equilibrium. If the characteristic equation of the linearized equation has a zero $\lambda \in (0, 1)$ we can assume that the difference equation has solutions with the following asymptotics

$$\bar{x} + \lambda^n + o(\lambda^{2n}). \tag{8}$$

At this point we require a result (such as Theorem C) which will guarantee the existence of solutions which have the needed asymptotics.

The following extension of Eq. (1)

$$x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} c_i x_{n-i}}, \quad n \in \mathbb{N}_0, \tag{9}$$

where $k \in \mathbb{N}$, $c_i \geq 0$, $i = 0, \ldots, k-1$, $\sum_{i=0}^{k-1} c_i = 1$, and $\alpha < -1$, was considered in [2]. Note that the linearized equation for Eq. (9) about the equilibrium $\bar{x} = \alpha + 1$ is

$$(\alpha + 1) y_{n+1} + \sum_{i=0}^{k-1} c_i y_{n-i} - y_{n-k} = 0. \tag{10}$$

For the case $\alpha > -1$ the characteristic polynomial associated with Eq. (10) i.e.,

$$P(t) = (\alpha + 1) t^{k+1} + \sum_{i=0}^{k-1} c_i t^{k-i} - 1 = 0, \tag{11}$$

has a positive root t_0 belonging to the interval (0, 1). To see this, note that $P(0) = -1$ and $P(1) = \alpha + 1 > 0$.

This fact motivated us to believe that in this case there are solutions of Eq. (9) which have the following asymptotics

$$x_n = \alpha + 1 + a t_0^n + o(t_0^n), \tag{12}$$

where $a \in \mathbb{R}$ and t_0 is the above-mentioned root of polynomial (11).

The problem of the existence of non-oscillatory solutions of Eq. (9) for the case $\alpha \leq -1$ is more interesting, since $\bar{x} = 0$ is not an equilibrium of Eq. (9). The fact makes it difficult to guess the asymptotics of the solutions which exist. In [2], for the case $\alpha < -1$, we expected that if such solutions exist they converge to zero geometrically, moreover we expect that for such solutions the first two members in their asymptotics are in the following form:

$$\varphi_n = a t^n + b t^{2n}, \tag{13}$$

where t is a number belonging to the interval (0, 1).

For the case of Eq. (9), t can be chosen in the following way. Since we consider only those solutions of Eq. (9) which are defined for all $n \in \mathbb{N}$, we can write the equation in the following form

$$(x_{n+1} - \alpha) \sum_{i=0}^{k-1} c_i x_{n-i} - x_{n-k} = 0. \tag{14}$$

For this equation, $\bar{x} = 0$ is an equilibrium and the corresponding linearized equation about the equilibrium $\bar{x} = 0$ is

$$\alpha \sum_{i=0}^{k-1} c_i x_{n-i} + x_{n-k} = 0. \tag{15}$$

Note that the characteristic equation

$$P_1(t) = \alpha \sum_{i=0}^{k-1} c_i t^{k-i} + 1$$

satisfies $P_1(0) = 1$ and $P_1(1) = 1 + \alpha < 0$ and that P_1 is decreasing on (0, 1). Hence P_1 has a unique characteristic zero $t = t_1$, which belongs to the interval (0, 1) if $\alpha < -1$. For such chosen t we have shown in [2] that Eq. (9) has solutions which have the first two members in their asymptotics as in (13).

3. Case $\alpha = -1$

Here, we consider the case $\alpha = -1$. Since in this case $P_1(1) = 0$ and $P_1(t) < 0$ when $t \in [0, 1)$, the polynomial P_1 does not have any zero in the interval (0, 1). So we came across a new problem, how to choose the asymptotics for possible solutions of Eq. (9) which converge to zero. In the following section among other results we describe how to choose the asymptotics for the case.

In this section we consider Eq. (1) for the case $\alpha = -1$, that is, we consider the equation

$$x_{n+1} = -1 + \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0. \tag{16}$$

First, we address Question 1, in the case. Note that Eq. (16) can be written in the following form

$$(x_{n+1}+1)x_n - x_{n-1} = 0.$$

The function $f(y,z) = y(z+1)$ is increasing in each variable if $y, z \in (0, \infty)$. Hence in view of Theorem C it is enough to find bounds (y_n) and (z_n) satisfying inequalities (7).

From our considerations in [36], we expect that a possible solution of Eq. (16) converging to zero might have the following asymptotics

$$\hat{x}_n = \frac{a}{n} + \frac{b\ln n + c}{n^2} + \frac{d\ln^2 n + e\ln n + f}{n^3}.$$

3.1. *Solution to* Question 1 *for the case* $\alpha = -1$

A solution to Question 1 gives the following result.

Theorem 1. *Eq.* (16) *has positive solutions converging to zero. Moreover, these solutions display the asymptotic behavior*

$$\frac{1}{n} + \frac{2\ln n + c}{n^2} + \frac{4\ln^2 n + (4c-4)\ln n}{n^3} + O\left(\frac{\ln n}{n^3}\right).$$

Proof. First, write Eq. (16) in the following form

$$F(x_{n-1}, x_n, x_{n+1}) = (x_{n+1}+1)x_n - x_{n-1} = 0. \tag{17}$$

Let φ_n be defined as follows

$$\varphi_n = \frac{a}{n} + \frac{b\ln n + c}{n^2} + \frac{d\ln^2 n + e\ln n + f}{n^3}.$$

Then, we have

$$\begin{aligned}
&F(\varphi_{n-1}, \varphi_n, \varphi_{n+1})\\
&= \left\{1 + \frac{a}{n}\left(1+\frac{1}{n}\right)^{-1} + \frac{b}{n^2}\left(1+\frac{1}{n}\right)^{-2}\left[\ln n + \ln\left(1+\frac{1}{n}\right)\right] + \frac{c}{n^2}\left(1+\frac{1}{n}\right)^{-2}\right.\\
&\quad \left. + \frac{d}{n^3}\left(1+\frac{1}{n}\right)^{-3}\left[\ln n + \ln\left(1+\frac{1}{n}\right)\right]^2 + \frac{e}{n^3}\left(1+\frac{1}{n}\right)^{-3}\left[\ln n + \ln\left(1+\frac{1}{n}\right)\right] + \frac{f}{n^3}\left(1+\frac{1}{n}\right)^{-3}\right\}\\
&\quad \times \left(\frac{a}{n} + b\frac{\ln n}{n^2} + \frac{c}{n^2} + d\frac{\ln^2 n}{n^3} + e\frac{\ln n}{n^3} + \frac{f}{n^3}\right)\\
&\quad - \left\{\frac{a}{n}\left(1-\frac{1}{n}\right)^{-1} + \frac{b}{n^2}\left(1-\frac{1}{n}\right)^{-2}\left[\ln n + \ln\left(1-\frac{1}{n}\right)\right] + \frac{c}{n^2}\left(1-\frac{1}{n}\right)^{-2}\right.\\
&\quad \left. + \frac{d}{n^3}\left(1-\frac{1}{n}\right)^{-3}\left[\ln n + \ln\left(1-\frac{1}{n}\right)\right]^2 + \frac{e}{n^3}\left(1-\frac{1}{n}\right)^{-3}\left[\ln n + \ln\left(1-\frac{1}{n}\right)\right] + \frac{f}{n^3}\left(1-\frac{1}{n}\right)^{-3}\right\}\\
&= \left(1 + \frac{a}{n} - \frac{a}{n^2} + \frac{a}{n^3} + b\frac{\ln n}{n^2} - 2b\frac{\ln n}{n^3} + 3b\frac{\ln n}{n^4} + \frac{b}{n^3} + \frac{c}{n^2} - \frac{2c}{n^3} + d\frac{\ln^2 n}{n^3} + e\frac{\ln n}{n^3} + O\left(\frac{1}{n^3}\right)\right)\\
&\quad \times \left(\frac{a}{n} + b\frac{\ln n}{n^2} + \frac{c}{n^2} + d\frac{\ln^2 n}{n^3} + e\frac{\ln n}{n^3} + \frac{f}{n^3}\right)\\
&\quad - \left(\frac{a}{n} + \frac{a}{n^2} + \frac{a}{n^3} + b\frac{\ln n}{n^2} + 2b\frac{\ln n}{n^3} + 3b\frac{\ln n}{n^4} - \frac{b}{n^3} + \frac{c}{n^2} + \frac{2c}{n^3} + d\frac{\ln^2 n}{n^3} - 2d\frac{\ln n}{n^4}\right.
\end{aligned}$$

$$+\ 3d\frac{\ln^2 n}{n^4}+e\frac{\ln n}{n^3}+3e\frac{\ln n}{n^4}+\frac{f}{n^3}+O\left(\frac{1}{n^4}\right)\Bigg)$$

$$=\Bigg(\frac{a}{n}+b\frac{\ln n}{n^2}+\frac{c}{n^2}+d\frac{\ln^2 n}{n^3}+e\frac{\ln n}{n^3}+\frac{f}{n^3}+\frac{a^2}{n^2}+ab\frac{\ln n}{n^3}+\frac{ac}{n^3}+ad\frac{\ln^2 n}{n^4}+ae\frac{\ln n}{n^4}-\frac{a^2}{n^3}$$

$$-ab\frac{\ln n}{n^4}+ab\frac{\ln n}{n^3}+b^2\frac{\ln^2 n}{n^4}-2ab\frac{\ln n}{n^4}+bc\frac{\ln n}{n^4}$$

$$+\ \frac{ac}{n^3}+bc\frac{\ln n}{n^4}+ad\frac{\ln^2 n}{n^4}+ae\frac{\ln n}{n^4}+O\left(\frac{1}{n^4}\right)\Bigg)$$

$$-\Bigg(\frac{a}{n}+\frac{a}{n^2}+\frac{a}{n^3}+b\frac{\ln n}{n^2}+2b\frac{\ln n}{n^3}-\frac{b}{n^3}+\frac{c}{n^2}+\frac{2c}{n^3}+d\frac{\ln^2 n}{n^3}+e\frac{\ln n}{n^3}+\frac{f}{n^3}$$

$$+\ 3d\frac{\ln^2 n}{n^4}+(3b-2d+3e)\frac{\ln n}{n^4}+O\left(\frac{1}{n^4}\right)\Bigg)$$

$$=\frac{a^2-a}{n^2}+2b(a-1)\frac{\ln n}{n^3}+\frac{2c(a-1)-a^2-a+b}{n^3}$$

$$+(2ad+b^2-3d)\frac{\ln^2 n}{n^4}+(2ae-3ab+2bc-3b+2d-3e)\frac{\ln n}{n^4}+O\left(\frac{1}{n^4}\right). \qquad (18)$$

Next, by equating the coefficients in (18) to zero, we find

$$a=1,\qquad b=2,\quad \text{and}\quad d=4.$$

Hence, by using these values of a, b and d, and replacing e by t into (18), we obtain

$$F(\varphi_{n-1},\varphi_n,\varphi_{n+1})=(\varphi_{n+1}+1)\varphi_n-\varphi_{n-1}=(4c-t-4)\frac{\ln n}{n^4}+O\left(\frac{1}{n^4}\right).$$

Fix $c\in\mathbb{R}$. It is clear that we can choose $t=e_1$ and $t=e_2$ so that

$$F=(4c-e_i-4)\frac{\ln n}{n^4}+O\left(\frac{1}{n^4}\right),\quad i=1,2,$$

is respectively, greater and less than zero, for sufficiently large n.

With the notation

$$y_n=\frac{a}{n}+\frac{b\ln n+c}{n^2}+\frac{d\ln^2 n+e_1\ln n}{n^3},$$
$$z_n=\frac{a}{n}+\frac{b\ln n+c}{n^2}+\frac{d\ln^2 n+e_2\ln n}{n^3},$$

we obtain

$$F(y_{n-1},y_n,y_{n+1})\sim(4c-e_1-4)\frac{\ln n}{n^4}>0$$

and

$$F(z_{n-1},z_n,z_{n+1})\sim(4c-e_2-4)\frac{\ln n}{n^4}<0.$$

These relations show that the inequalities (5) are satisfied for sufficiently large n, where $k=1$, $f(x_{n-1},x_n,x_{n+1})=F(x_{n-1},x_n,x_{n+1})+x_{n-1}$, and F is given by (17). Hence, by Theorem C it follows that there are solutions (x_n) of Eq. (16) with the following asymptotics

$$x_n=\frac{1}{n}+\frac{2\ln n+c}{n^2}+\frac{4\ln^2 n+(4c-4)\ln n}{n^3}+O\left(\frac{\ln n}{n^3}\right)$$

and consequently Eq. (16) has solutions converging to zero. □

Similarly, the following extension of Theorem 1 can be proved.

Theorem 1(a). *Assume that $\alpha = -1$. Then there is a positive solution of Eq. (9) converging to zero.*

Remark 2. The existence of solutions of Eq. (1) converging to zero, when $\alpha \le -1$, was also proved in [8] by a quite different method. However, the method in the present paper based on an inclusion theorem enables us to find the asymptotics of such solutions, unlike the method in [8].

Since we have showed that there are positive solutions of Eq. (9) converging to zero, it is natural to investigate their behavior in detail. Before this we prove a very important and quite general result for a class of second-order difference equations with a two-periodic coefficient, which is a natural extension of Theorem 2 in [31].

Theorem 2. *Assume that α_n is two-periodic sequence, f and g are non-decreasing continuous functions which map the interval $(0, \infty)$ into itself, and that (x_n) is a positive solution of the difference equation*

$$x_n = \alpha_n + \frac{f(x_{n-2})}{g(x_{n-1})}. \tag{19}$$

Then, the sequences (x_{2n}) and (x_{2n+1}) are eventually monotone.

Proof. From (19), and since α_n is two-periodic sequence, we have that

$$x_n - x_{n-2} = \frac{(f(x_{n-2}) - f(x_{n-4}))g(x_{n-3}) + f(x_{n-4})(g(x_{n-3}) - g(x_{n-1}))}{g(x_{n-1})g(x_{n-3})},$$

from which it follows that for $n \ge 1$

$$x_{2n+1} - x_{2n-1} = \frac{(f(x_{2n-1}) - f(x_{2n-3}))g(x_{2n-2}) + f(x_{2n-3})(g(x_{2n-2}) - g(x_{2n}))}{g(x_{2n})g(x_{2n-2})} \tag{20}$$

and

$$x_{2n+2} - x_{2n} = \frac{(f(x_{2n}) - f(x_{2n-2}))g(x_{2n-1}) + f(x_{2n-2})(g(x_{2n-1}) - g(x_{2n+1}))}{g(x_{2n+1})g(x_{2n-1})}. \tag{21}$$

Case 1. If $x_1 \ge x_{-1}$ and $x_0 \ge x_2$ from (20) we obtain $x_3 \ge x_1$. From this and (21) it follows that $x_4 \le x_2$. By induction we obtain

$$x_0 \ge x_2 \ge \cdots \ge x_{2n} \ge \cdots \quad \text{and} \quad \cdots \ge x_{2n+1} \ge x_{2n-1} \ge \cdots \ge x_1 \ge x_{-1}.$$

Case 2. The case $x_1 \le x_{-1}$ and $x_0 \le x_2$ can be treated similarly to Case 1. Hence we omit the proof in the case.

Case 3. Assume that $x_{-1} \le x_1$ and $x_0 \le x_2$. If $x_1 \ge x_3$, then by similar arguments to those in Case 1 it can be obtained that

$$x_0 \le x_2 \le \cdots \le x_{2n} \le \cdots \quad \text{and} \quad \cdots \le x_{2n+1} \le x_{2n-1} \le \cdots \le x_3 \le x_1.$$

Hence, we may assume that $x_{-1} \le x_1 \le x_3$ and $x_0 \le x_2$. If further $x_4 \le x_2$, then similarly we have that

$$x_2 \ge x_4 \ge \cdots \ge x_{2n} \ge \cdots \quad \text{and} \quad \cdots \ge x_{2n+1} \ge x_{2n-1} \ge \cdots \ge x_1 \ge x_{-1}.$$

So we may assume that $x_{-1} \le x_1 \le x_3$ and $x_0 \le x_2 \le x_4$.

Repeating this procedure we have that, there is a $k \in \mathbb{N}$ such that

$$x_0 \le x_2 \le \cdots \le x_{2k}, \qquad x_{-1} \le x_1 \le \cdots \le x_{2k-3} \le x_{2k-1}, \qquad x_{2k+1} \le x_{2k-1}, \tag{22}$$

or

$$x_0 \le x_2 \le \cdots \le x_{2k}, \qquad x_{-1} \le x_1 \le \cdots \le x_{2k-1} \le x_{2k+1}, \qquad x_{2k+2} \le x_{2k}, \tag{23}$$

or there is no such $k \in \mathbb{N}$, that is,

$$x_0 \le x_2 \le \cdots \le x_{2n} \le \cdots \quad \text{and} \quad x_{-1} \le x_1 \le \cdots \le x_{2k-1} \le x_{2k+1} \le \cdots,$$

which means that the sequences (x_{2n}) and (x_{2n+1}) are monotonous.

If (22) holds, then by (20) and (21) we have that

$$x_0 \le x_2 \le \cdots \le x_{2n} \le \cdots \quad \text{and} \quad \cdots \le x_{2(n+k)+1} \le x_{2(n+k)-1} \le \cdots \le x_{2k+1} \le x_{2k-1}.$$

On the other hand, if (23) holds, then by (20) and (21) we have that

$$\cdots \le x_{2(n+k)} \cdots \le x_{2k+2} \le x_{2k} \quad \text{and} \quad x_{-1} \le x_1 \le \cdots \le x_{2k-1} \le x_{2k+1} \le \cdots.$$

From all above mentioned the result follows, in this case.

Case 4. The case $x_1 \le x_{-1}$ and $x_0 \ge x_2$ can be treated similarly to Case 3, so it will be omitted. □

Theorem 3. *Every positive solution of Eq.* (16) *converges to zero.*

Proof. From

$$x_{n+1} = \frac{x_{n-1} - x_n}{x_n} > 0, \quad n \in \mathbb{N}_0,$$

it follows that the sequence x_n is decreasing and $x_n > 0$, $n \ge -1$. Hence, there is $\lim_{n\to\infty} x_n = x \in [0, \infty)$. If $x > 0$, then letting $n \to \infty$ in (16), we obtain that $x = 0$, which is a contradiction. Hence $x = 0$, as desired. □

4. Case $\alpha = -2$

In this section we consider the equation

$$x_{n+1} = -2 + \frac{x_{n-1}}{x_n}, \quad n \in \mathbb{N}_0. \tag{24}$$

The linearized equation associated with Eq. (24) about the equilibrium $\bar{x} = -1$, is

$$y_{n+1} - y_n + y_{n-1} = 0, \quad n \in \mathbb{N}_0.$$

Its characteristic roots are

$$\lambda_{1,2} = \frac{1 \pm \mathrm{i}\sqrt{3}}{2}.$$

Hence, the linearized stability theorem fails for this case ([20, p. 14]).

Concerning Question 5, we are not able, at the moment, to solve the stability of the equilibrium point $\bar{x} = -1$ of Eq. (24), however we offer the following conjecture.

Conjecture 1. *The equilibrium point $\bar{x} = -1$ of Eq.* (24) *is not stable.*

4.1. Periodic character of Eq. (24)

Assume first that

$$\ldots, a, b, a, b, \ldots$$

is a two-periodic solution of Eq. (24). Then it must be that

$$a = -2 + \frac{a}{b} \quad \text{and} \quad b = -2 + \frac{b}{a}$$

from which it follows that $a = b = -1$. Hence, Eq. (24) does not have prime two-periodic solutions.

For the case of prime three-periodic solutions, we have the following result.

Theorem 4. *Eq.* (24) *has a prime three-periodic solution.*

Proof. Assume that

$$\ldots, a, b, c, a, b, c, \ldots$$

is a prime three-periodic solution. Then it must be that

$$bc = -2b + a, \qquad ac = -2c + b, \qquad ab = -2a + c. \tag{25}$$

Hence

$$ab + bc + ac = -(a + b + c) = p.$$

From (25) we have that

$$abc = -2ab + a^2 = -2bc + b^2 = -2ac + c^2,$$

which implies that

$$3abc = (a + b + c)^2 - 4(ab + bc + ac) = p^2 - 4p.$$

Also, squaring equalities (25) and summing such obtained expressions, we obtain

$$a^2b^2 + b^2c^2 + c^2a^2 = 5(a^2 + b^2 + c^2) - 4(ab + bc + ac) = 5p^2 - 14p.$$

On the other hand, we have

$$a^2b^2 + b^2c^2 + c^2a^2 = (ab + bc + ac)^2 - 2abc(a + b + c) = p^2 + \frac{2}{3}p(p^2 - 4p),$$

from which it follows that

$$2p(p^2 - 10p + 21) = 0.$$

Hence, $p = 3$ or $p = 7$.

Now note that a, b, c are roots of the polynomial $P_3(z) = (z - a)(z - b)(z - c)$. If $p = 3$ then a, b, c are roots of the polynomial $(z + 1)^3 = 0$, so that $a = b = c = -1$. Hence $p = 7$, and a, b, c are real roots of the polynomial

$$P_3(z) = z^3 + 7z^2 + 7z - 7,$$

finishing the proof of the theorem. □

5. Case $\alpha < -1$

Here we give an answer to Question 3. As we have already mentioned in the introduction, in [12, Lemma 4.1] it was proved that if $\alpha < -1$ (actually, the proof holds for $\alpha \le -1$) and if for a nontrivial solution (x_n) of Eq. (1) there is an $n_0 \in \mathbb{N}$ such that $x_n \le \alpha + 1$ for every $n \ge n_0$ then it decreases to $-\infty$.

Here, we show that such a solution of Eq. (1) does not exist by proving the following result.

Lemma 1. *Assume that $\alpha < 0$. Then, there are no solutions of Eq. (1) which eventually decrease to $-\infty$.*

Proof. Assume that (x_n) is a solution to Eq. (1) which eventually decreases to $-\infty$. Then it must be

$$\alpha + \frac{x_{n-1}}{x_n} = x_{n+1} < x_n < 0,$$

from which it follows that

$$\alpha x_n + x_{n-1} > x_n^2$$

and consequently

$$|\alpha||x_n| = \alpha x_n > x_n^2 \Leftrightarrow |x_n| < |\alpha|,$$

hence the sequence (x_n) must be bounded, which is a contradiction. □

Theorem 5. *Assume that $\alpha < 0$. Then, there are no negative decreasing solutions of Eq. (1).*

Proof. According to Lemma 1 we have that each eventually negative decreasing solution (x_n) of Eq. (1) is convergent. Hence, $\lim_{n\to\infty} x_n = \alpha + 1$, and $x_n \geq \alpha + 1$, for sufficiently large n. Clearly, this is impossible if $\alpha \in [-1, 0)$. Hence, assume that $\alpha < -1$. However, if it were, by Theorem A, it would be that x_n monotonically converges to zero, which would imply that x_n is nonnegative, arriving at a contradiction. □

For the case $\alpha \in [-1, 0)$ we have the following stronger result.

Theorem 6. *Assume that $\alpha \in [-1, 0)$. Then there are no negative solutions of Eq.* (1).

Proof. First, assume that (x_n) is a negative solution of Eq. (1) and that $\alpha \in (-1, 0)$. We have that

$$\alpha + \frac{x_{n-1}}{x_n} = x_{n+1} < 0, \quad n \in \mathbb{N},$$

from which it follows that

$$|x_n| \geq \frac{1}{|\alpha|}|x_{n-1}| > |x_{n-1}|. \tag{26}$$

Letting $n \to \infty$ in (26) we have that the sequence $|x_n|$ increasingly tends to $+\infty$, that is, x_n decreasingly tends to $-\infty$, which contradicts Theorem 5.

If $\alpha = -1$, then

$$x_{n+1} = \frac{x_{n-1} - x_n}{x_n} < 0, \quad n \in \mathbb{N},$$

which implies that $x_{n-1} - x_n > 0$, that is, x_n is decreasing, which is impossible by Theorem 5. □

Question 6. Are there negative solutions of Eq. (1) for the case $\alpha < -1$?

6. Invariant intervals and global convergence

This section is devoted to finding more natural invariant intervals of Eq. (1), than those in Theorem 3.2 in [12]. In particular, we find an invariant interval of type $[-M_\alpha, -m_\alpha]$, where $0 < m_\alpha < M_\alpha$, for the function

$$f(x, y) = \alpha + \frac{y}{x}. \tag{27}$$

Theorem 7. *Assume that $\alpha < -3$. Then there are positive constants m_α and M_α, $1 < m_\alpha < M_a < \infty$, such that the interval $[-M_\alpha, -m_\alpha]$ is invariant for the function* (27).

Proof. Since the function f is decreasing in y for fixed negative x and increasing in x for fixed negative y, we try to find m_α and M_α such that

$$-M_\alpha \leq \alpha + \frac{-m_\alpha}{-M_\alpha} \leq \alpha + \frac{y}{x} \leq \alpha + \frac{-M_\alpha}{-m_\alpha} \leq -m_\alpha,$$

for every $x, y \in [-M_\alpha, -m_\alpha]$.

Assume that $m_\alpha \neq M_\alpha$ satisfy the system

$$-M_\alpha = \alpha + \frac{-m_\alpha}{-M_\alpha} \quad \text{and} \quad \alpha + \frac{-M_\alpha}{-m_\alpha} = -m_\alpha. \tag{28}$$

Then we have

$$1 = \frac{1}{m_\alpha} + \frac{1}{M_\alpha}. \tag{29}$$

Note that (29) implies that m_α and M_α are both greater than 1. Substituting (29) in (28), we obtain that the equation

$$f_\alpha(x) = (\alpha + x)\left(\alpha + \frac{x}{x-1}\right) - 1, \tag{30}$$

must have a root, for $x > 1$. Moreover, notice that $f_\alpha(m_\alpha) = f_\alpha(M_\alpha) = 0$. In fact, f_α has exactly two roots whenever $\alpha < -3$, since $\lim_{x\to 1+0} f_\alpha(x) = \lim_{x\to+\infty} f_\alpha(x) = -\infty$, it is increasing on the interval $(1, 2)$, decreasing on the interval $(2, \infty)$ and consequently it attains its maximum on $(1, \infty)$ at the point $x = 2$ having the value $f_\alpha(2) = \alpha^2 + 4\alpha + 3 = (\alpha + 2)^2 - 1 > 0$, for $\alpha < -3$.

From (30) by direct calculation, we obtain

$$m_\alpha = \frac{1-\alpha-\sqrt{\alpha^2+2\alpha-3}}{2}, \qquad M_\alpha = \frac{1-\alpha+\sqrt{\alpha^2+2\alpha-3}}{2}$$

which are positive numbers when $\alpha < -3$, finishing the proof of the theorem. □

Remark 3. By the homogenicity of $f(x, y)$ it follows that for each $k \in \mathbb{N}$ the interval $[-kM_\alpha, -km_\alpha]$ is mapped by the function $f(x, y)$ into the interval $[-M_\alpha, -m_\alpha]$, hence the set

$$\bigcup_{k=1}^{\infty}[-kM_\alpha, -km_\alpha],$$

is mapped by f into $[-M_\alpha, -m_\alpha]$.

As we said in the introduction Hamza in [12] wrongly applied Theorem 1.4.6 in [22]. Instead of the theorem he should have applied the following theorem (see, [22, Theorem 1.4.5]):

Theorem D. *Let $[a, b]$ be an invariant interval of real numbers and assume that*

$$f : [a, b]^2 \to [a, b]$$

is a continuous function satisfying the following conditions:

(a) *$f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$.*
(b) *If $(m, M) \in [a, b]^2$ is a solution of the system*

$$m = f(m, M) \quad \text{and} \quad M = f(M, m), \tag{31}$$

then $M = m$.

Then Eq. (3) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of Eq. (3) converges to $\bar{x}$.

Now we prove a global convergence result concerning (1) for the case $\alpha < -5$. It is interesting that by Theorems 7 and D it is not immediately clear that $\bar{x} = \alpha+1$ is a global attractor of Eq. (1) relative to the set $\bigcup_{k=1}^{\infty}[-kM_\alpha, -km_\alpha]$. The problem is that we cannot prove that system (31) has only the equilibrium solution (notice that $-m_\alpha$ and $-M_\alpha$ in Theorem 7 verify this system).

Theorem 8. *Assume that $\delta > 0$, $\gamma > 1$ and*

$$\alpha \in \left(-\infty, \min\left\{-(1+2\gamma), -\frac{\gamma^2+\delta}{\gamma-1}\right\}\right]. \tag{32}$$

Then $\bar{x} = \alpha + 1$ is a global attractor of Eq. (1) relative to the set $[\alpha - \delta, \alpha + \gamma]^2$ (i.e. every solution with initial conditions $x_{-1}, x_0 \in [\alpha - \delta, \alpha + \gamma]$ converges to $\bar{x} = \alpha + 1$).

Proof. First we prove that the interval $[\alpha - \delta, \alpha + \gamma]$ is invariant for the function

$$f(x, y) = \alpha + \frac{y}{x}.$$

Since $\alpha < -(\gamma^2 + \delta)/(\gamma - 1)$ and $\alpha + \gamma < -1$, we have

$$f(x, y) \le f(\alpha+\gamma, \alpha-\delta) = \alpha + \frac{\alpha-\delta}{\alpha+\gamma} \le \alpha + \gamma.$$

On the other hand, it is clear that

$$f(x, y) \ge f(\alpha-\delta, \alpha+\gamma) = \alpha + \frac{\alpha+\gamma}{\alpha-\delta} > \alpha > \alpha - \delta,$$

from which the invariance of the interval $[\alpha - \delta, \alpha + \gamma]$ follows.

Since f satisfies Condition (a) of Theorem D, we only need to prove that the system

$$m = \alpha + \frac{M}{m} \quad \text{and} \quad M = \alpha + \frac{m}{M} \tag{33}$$

has a unique solution $m = M = \alpha + 1$. From (33) we have that

$$m^2 - \alpha m - M = 0 \quad \text{and} \quad M^2 - \alpha M - m = 0.$$

Subtracting these two expressions it follows that

$$(M - m)(M + m - \alpha + 1) = 0.$$

Since $m, M \leq \alpha + \gamma$ and from (32) it follows that

$$M + m - \alpha + 1 \leq \alpha + 1 + 2\gamma < 0.$$

Hence, $m = M$ from which the result follows. □

7. Applications

In [23] Ladas posed the following conjecture.

Conjecture 2. *Every solution of the equation*

$$x_{n+1} = \frac{x_n - 1}{x_n - x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{34}$$

converges to the two cycle

$$\ldots, 1, 0, 1, 0, \ldots.$$

By the change

$$z_n = \frac{1}{x_n - 1}$$

Eq. (34) becomes

$$z_{n+1} = -1 + \frac{z_{n-1}}{z_n}, \quad n \in \mathbb{N}_0. \tag{35}$$

By Theorem 1, it follows that there is a positive solution of Eq. (35) converging to zero, which implies that there is a solution of Eq. (34) diverging to $+\infty$, so that Conjecture 2 is refuted. Note that we also know that the solution has the following asymptotics

$$x_n = 1 + \frac{1}{z_n} = n - 2\ln n + 1 - c + o(1).$$

Remark 4. Note that the following equation from Open Problem 6.10.7 in [22]

$$y_{n+1} = \frac{1 - y_{n-1}}{1 - y_n}, \quad n \in \mathbb{N}_0, \tag{36}$$

can be reduced to Eq. (34) by the change $x_n = y_n - 1$. Thus, all the results of this paper, for the case $\alpha = -1$, can be applied to Eq. (36).

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